Sehldon ross a first course in probability 10th ed 2018 page 438/848

Assume that the functions g_1 and g_2 satisfy the following conditions:

1. The equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2, with solutions given by, say, $x_1 = h_1(y_1, y_2), x_2 = h_2(y_1, y_2)$.
2. The functions g_1 and g_2 have continuous partial derivatives at all points (x_1, x_2) and are such that the 2×2 determinant

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} \equiv \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} \neq 0$$

   at all points (x_1, x_2).

Under these two conditions, it can be shown that the random variables Y_1 and Y_2 are jointly continuous with joint density function given by

(7.1)

$$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1, x_2)\left|J(x_1, x_2)\right|^{-1}$$

where $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$.

A proof of **Equation (7.1)** would proceed along the following lines:

(7.2)

$$P\{Y_1 \le y_1, Y_2 \le y_2\} = \iint\limits_{\substack{(x_1, x_2): \\ g_1(x_1, x_2) \le y_1 \\ g_2(x_1, x_2) \le y_2}} f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

The joint density function can now be obtained by differentiating **Equation (7.2)** with respect to y_1 and y_2. That the result of this differentiation will be equal to the right-hand side of **Equation (7.1)** is an exercise in advanced calculus whose proof will not be presented in this book.

Example 7a

Let X_1 and X_2 be jointly continuous random variables with probability density function f_{X_1,X_2}. Let $Y_1 = X_1 + X_2, Y_2 = X_1 - X_2$. Find the joint density function of Y_1 and Y_2 in terms of f_{X_1,X_2}.

Solution

Let $g_1(x_1, x_2) = x_1 + x_2$ and $g_2(x_1, x_2) = x_1 - x_2$. Then

$$J(x_1, x_2) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \qquad |J|^{-1} = \frac{1}{2}$$

Also, since the equations $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$ have $x_1 = (y_1 + y_2)/2$, $x_2 = (y_1 - y_2)/2$ as their solution, it follows from **Equation (7.1)** that the desired density is

$0 \le x_1 \le 1$ A

$\to 0 \le x_2 \le 1 \to 0 \le (y_1 - y_2) \le 2$ B

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2} f_{X_1,X_2}\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right)$$

For instance, if X_1 and X_2 are independent uniform (0, 1) random variables, then

A B

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{1}{2} & 0 \le y_1 + y_2 \le 2,\ 0 \le y_1 - y_2 \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$X_1 \sim \lambda_1 e^{-\lambda_1 x}$

$X_2 \sim \lambda_2 e^{-\lambda_2 x}$

or if X_1 and X_2 are independent exponential random variables with respective parameters λ_1 and λ_2, then

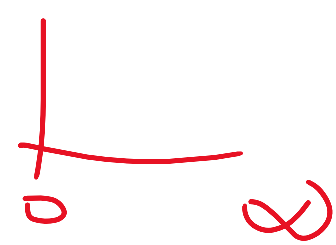
Suppose that the lifetimes of two components are independent of one another and that the first lifetime, X_1, has an **exponential distribution** with parameter λ_1 whereas the second, X_2, has an **exponential distribution** with parameter λ_2. Then the joint pdf is

$\lambda_1 \lambda_2 e^{-\lambda_1 x_1 - \lambda_2 x_2}$

$$f(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)$$

$$= \begin{cases} \lambda_1 e^{-\lambda_1 x_1} \cdot \lambda_2 e^{-\lambda_2 x_2} & x_1 \geq 0, x_2 \geq 0 \\ 0 & \textit{otherwise} \end{cases}$$

A B



$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{\lambda_1 \lambda_2}{2} \exp\left\{-\lambda_1\left(\frac{y_1 + y_2}{2}\right) - \lambda_2\left(\frac{y_1 - y_2}{2}\right)\right\} & y_1 + y_2 \ge 0,\ y_1 - y_2 \ge 0 \\ & \text{otherwise} \end{cases}$$

$$g(x_1, x_2) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(x_1^2 + x_2^2)\right]$$

$x_1 = \frac{y_1 + y_2}{2}$ $\quad$ $x_2 = \frac{y_1 - y_2}{2}$

Finally, if X_1 and X_2 are independent standard normal random variables, then

$\frac{1}{|J| 2\pi}$

$2 * 2\pi$

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= \frac{1}{4\pi} e^{-[(y_1+y_2)^2/8 + (y_1-y_2)^2/8]} \\ &= \frac{1}{4\pi} e^{-(y^2{}_1 + y^2{}_2)/4} \\ &= \frac{1}{\sqrt{4\pi}} e^{-y^2{}_1/4} \frac{1}{\sqrt{4\pi}} e^{-y^2{}_2/4} \end{aligned}$$

Thus, not only do we obtain (in agreement with **Proposition 3.2**) that both $X_1 + X_2$ and $X_1 - X_2$ are normal with mean 0 and variance 2, but we also conclude that these two random variables are independent. (In fact, it can be shown that if X_1 and X_2 are independent random variables having a common distribution function F, then $X_1 + X_2$ will be independent of $X_1 - X_2$ if and only if F is a normal distribution function.)